SUMMARY OF FORMULAS FOR FLAT PLATES

OF PLYWOOD UNDER UNIFORM OR

CONCENTRATED LOADS

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Foreword

A basic study of plywood that is under way at the Forest
Products Laboratory has included as one phase the mathematical anal-
ysis of the deflection of flat plates under uniformly distributed
or concentrated loads. This theoretical analysis has progressed
sufficiently to permit the publication of the formulas presented
in this mimeograph. Some of these formulas have been checked
against test results, and the others are believed to afford
reasonably accurate results.

Other phases of the study of plywood relate to basic
strength in compression, tension, bending, and shear; resistance
to combined stress; criterion for buckling in flat and curved
plates and shells and behavior after buckling; and methods of re-
inforcing. It is planned that as rapidly as significant results
become available, they will be presented in this series of reports.

Forest Products Laboratory

SUMMARY OF FORMULAS FOR FLAT PLATES OF PLYWOOD
UNDER UNIFORM OR CONCENTRATED LOADS[1]

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The material herewith presented comprises a summary of the principal results of a more extensive report soon to be issued by the Forest Products Laboratory. Reference should be made to the extensive report for the derivation and discussion of the results contained in this summary.

Rectangular plywood plates will be considered in which the directions of the grain of the wood in adjacent plies are mutually perpendicular, and perpendicular or parallel to the edges of the plate. The plies are assumed to be either flat grain or edge grain. The choice of axes is shown in figure 1.

The effect of the glue other than that of securing adherence of adjacent plies is assumed to be negligible. Consequently, the formulas and methods of this summary are not intended to apply directly to partially or completely impregnated plywood or compregnated wood, although it is to be expected that many of the results of the extensive report apply to such material.

Notation
a = width of plate
b = length of plate
h = thickness of plate
w_0 = deflection at center of plate
p = load per unit area
P = concentrated load (section 5)
$P = pa^4/E_1 h^4$ (sections 6 and 7)
W = deflection at center of an infinitely long plate of width a under a specified type of load
$B = b\left(\dfrac{E_1}{E_2}\right)^{1/4}$ where E_1 and E_2 are defined in section 1

1. Stiffness in bending of strips of plywood

Consider a strip of plywood with its edges either parallel or perpendicular to the grain of the face plies and denote by x the direction parallel to the length of the strip. The stiffness of the strip is

determined by a modulus E_1 defined by the equation

$$E_1 I = \sum (E_x)_i I_i$$

where the summation is extended over all of the plies numbered, for example, as in figure 2; $(E_x)_i$ is the Young's modulus of the ith ply measured in a direction parallel to the length of the strip; I_i is the moment of inertia, with respect to the neutral axis, of the area of the cross section of the ith ply made by a plane perpendicular to the length of the strip; and I is the moment of inertia of the entire cross section of the strip with respect to its central line, that is, $I = h^3/12$ for a strip of unit width. An approximate formula in which the error is very slight is obtained for $E_1 I$ by taking the sum of the products $(E_x)_i I_i$ formed for only those plies in which the grain is parallel to the length of the strip. Exception is to be made of a three-ply strip having the grain of the face plies perpendicular to the length of the strip.

In the case of a rectangular plate with sides a and b, the modulus E_1 would determine the stiffness of a strip cut from the plate with its edges parallel to the side a as in figure 1. The modulus E_2 similarly defined, namely,

$$E_2 I = \sum (E_y)_i I_i$$

determines the stiffness of strips parallel to the side b.

As in the case of E_1 the calculation of E_2 can be based on the parallel plies only, except in the case of a three-ply strip having the grain of the face plies perpendicular to the length of the strip.

2. Young's modulus of a strip of
   plywood in tension or compression

As in section 1, consider a strip of plywood whose edges are parallel to the X-axis or to the side a of the rectangular plate of figure 1.

The mean modulus E_a in tension or compression may be defined by the equation

$$E_a h = \sum (E_x)_i h_i$$

where $(E_x)_i$ has the same meaning as in section 1, h_i is the thickness of the ith ply and h is the thickness of the strip.

In like manner for a strip parallel to the side b

$$E_b h = \sum (E_y)_i h_i$$

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The moduli E_a and E_b are needed in cases where the deflections of the plates under consideration are so large that direct stresses in addition to bending stresses are developed. These moduli can be calculated with little error by considering only those plies which are parallel to the length of the strip.

3. Rectangular plate. Uniformly distributed load. Edges simply supported. Small deflections

The method presented below may be taken to apply if the loads are such that the deflections do not exceed the thickness of the plate. Appreciable direct stresses develop at deflections of the order of magnitude of the thickness of the plate and the deflection will be less than that found by the method presented. For a plate whose length exceeds its breadth by a moderate amount the method of section 6 can be applied in this case.

It is assumed throughout that the corners of the plate are held down.

To find the deflection w_0 at the center of a given plate of width a and length b, calculate first the central deflection W of a similarly loaded very long plate (infinite strip) of width a and of the same construction. Now

$$W = \frac{5 \times 0.99}{32} \frac{pa^4}{E_1 h^3} = 0.1547 \frac{pa^4}{E_1 h^3} \tag{1}$$

Except for the factor 0.99, which expresses the plate effect (in wood practically negligible), this is the formula for the central deflection of a beam of unit width under a uniformly distributed load.

Then the deflection at the center of the given plate can be found approximately from the formula

$$w_0 = f \, W \tag{2}$$

where f is a factor to be taken from the curve of figure 3 corresponding to the argument

$$\frac{B}{a} = \frac{b}{a}\left(\frac{E_1}{E_2}\right)^{1/4} \tag{3}$$

The points shown in figure 3 were determined by an exact method, using the elastic constants of spruce, for various types of plywood. The curve is merely a smooth average curve determined by these points. A

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consideration of the extended analysis discloses the fact that the essential factors determining the central deflection of a plate under the conditions of loading and support of this section are the two moduli E_1 and E_2 that enter into the determination of W and B. Variations in other elastic constants will account for variations of the order of magnitude shown by the points in figure 3.[2] Hence it appears that this curve may be used for plywood of the type described at the beginning of this summary, independently of the species of wood used. The constants E_1 and E_2 must be known. They can be determined by calculation or by static bending tests of strips of matched material.

The maximum shown in figure 3 in the vicinity of $B/a = 2$, which at first sight appears to be impossible, is found in the exact analysis. It is associated with a wave form of slight amplitude that is assumed by the deflected surface of a plywood plate.

A presentation of the results of an approximate analysis in essentially the form (2) was made by C. B. Norris.[3] Because of the approximations involved, the deflections calculated from his results are too small, a fact which he recognized would be the case.

4. Rectangular plate. Uniformly distributed load. Edges clamped. Small deflections

In this case the central deflection of the corresponding infinite strip is given by

$$W = \frac{0.99}{32} \frac{pa^4}{E_1 h^3} = 0.0309 \frac{pa^4}{E_1 h^3} \tag{4}$$

The deflection of a finite plate can be found from the formula

$$w_0 = f W \tag{5}$$

in which f is to be taken from the curve of figure 4 where it is shown as a function of B/a, B being related to b by (3). The points shown near the curve in figure 4 are the exact values of f for various types of plywood. The elastic constants of spruce were used, but the curve may be used for wood of other species as pointed out in section 3.

The actual deflections will usually be considerably larger than those calculated by (5) because perfect clamping of the edges is rarely realized in practice. If the edges are restrained from moving inward, direct stresses will develop at moderate deflections. In this case the methods of section 7 are available for a plate whose length is moderately greater than its breadth.

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[2] It was convenient to calculate the points shown in some of the figures for plates in which the plies are all of the same thickness but the formulas and curves of this report are not restricted to such plates.

[3] Hardwood Record, May 1937.

5. Rectangular plate. Load concentrated at the center. Edges simply supported. Small deflections

The central deflection of the corresponding infinite strip is given by

$$\overline{w} = \frac{1.051 \times 6 \times 0.99}{0.8 \ \pi^3} \left(\frac{E_1}{E_2} \right)^{1/4} \frac{Pa^2}{E_1 h^3} \tag{6}$$

$$= 0.252 \left(\frac{E_1}{E_2} \right)^{1/4} \frac{Pa^2}{E_1 h^3} \tag{7}$$

The central deflection of a finite plate can be found from the formula

$$w_o = f \ \overline{w} \tag{8}$$

where f is to be taken from the curve of figure 5. In formula (6) the number 0.8 is an approximate figure for a constant whose values may range from 0.76 to 0.86, for various types of ordinary plywood. A means of calculating this constant is to be found in the extended report.

6. Infinite strip. (Long, narrow, rectangular plate). Uniformly distributed load. Edges simply supported

The formulas of this and the next section are applicable when the maximum deflection is small in comparison with the width of the strip, although possibly equal to several times the thickness of the strip.

In addition to the conditions stated in the heading, the edges are assumed to be restrained from moving inward.

Using the notation

$$P = \frac{pa^4}{E_1 h^4} \tag{9}$$

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the following approximate relation connects the load and the central deflection W.

$$P = H \left(\frac{W}{h} \right) + K \left(\frac{W}{h} \right)^3 \tag{10}$$

where

$$H = 6.46 \ E_1/E_L \tag{11}$$

$$K = 20.8 \ E_a/E_L \tag{12}$$

where E_L denotes Young's modulus in the direction of the grain of the wood. If the plywood is made of wood of more than one species, equation (10) can be multiplied through by E_L. There results a relation in which only the moduli E_1 and E_a enter.

The mean direct stress is given by the formula

$$g = 2.60 \ E_a \left(\frac{h}{a} \right)^2 \left(\frac{W}{h} \right)^2 \tag{13}$$

This is the mean direct stress averaged over the thickness of the plate. The direct stress in any ply can be calculated by observing that the stress in any ply is proportional to the E of that ply in a direction parallel to the X-axis. This follows from the fact that the strain associated with the direct stress is constant across the thickness of the plate.

The maximum bending stress in a face ply can be calculated by the approximate formula

$$s = 1.01 \ \alpha \ E_x \left(\frac{h}{a} \right)^2 \frac{W}{h} \tag{14}$$

where α is to be taken from the curve of figure 6. In this figure the argument $\mathcal{n}$ is connected with the deflection by the formula

$$\mathcal{n} = 2.778 \left(\frac{E_a}{E_1} \right)^{1/2} \frac{W}{h} . \tag{15}$$

The bending stress at any point in any other ply can be calculated by noting that the associated strain varies linearly with the distance from the neutral plane and that the corresponding stress is the product of this strain and E_x at the point under consideration. The formulas just given can be used for a plate whose length exceeds its breadth by a moderate amount and for small as well as large deflections. An inspection of figure 3 indicates that these formulas

can be used with small error if $\dfrac{B}{a} = \dfrac{b}{a} \left(\dfrac{E_1}{E_2} \right)^{1/4}$ is greater than

1.75. It is to be expected that the stresses calculated in this way will be satisfactory approximations to the stresses in the central portion of such a plate.

7. Infinite strip. (Long, narrow plate). Uniformly distributed load. Edges clamped

The edges are further assumed to be restrained from moving inward.

Using the notation

$$P = \frac{pa^4}{E_L h^4}$$

the following approximate equation holds

$$P = H \left(\frac{W}{h}\right) + K \left(\frac{W}{h}\right)^3 \tag{16}$$

where

$$H = 32.3 \frac{E_1}{E_L} \tag{17}$$

$$K = 23.2 \frac{E_a}{E_L} \tag{18}$$

Equation (16) may be multiplied through by E_L and thus be made applicable if the plywood is made of wood of two or more different species. [See discussion following (10)].

The mean direct stress is obtained from the approximate formula

$$g = 2.51 \, E_a \left(\frac{h}{a}\right)^2 \left(\frac{W}{h}\right)^2 \tag{19}$$

and the maximum bending stress in a face ply from the approximate formula

$$S = 1.01 \, \alpha \, E_x \left(\frac{h}{a}\right)^2 \left(\frac{W}{h}\right) \tag{20}$$

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where α is to be taken from the curve of figure 7. In this curve, the argument η is connected with the deflection by the formula

$$\eta = 2.732 \left(\frac{E_a}{E_1}\right)^{1/2} \frac{W}{h} \qquad (21)$$

An inspection of figure 4 justifies the conclusion that the formulas just written can be used to calculate approximately the central deflection and the stresses in the central portion of a plate for which

$$\frac{B}{a} = \frac{b}{a} \left(\frac{E_1}{E_2}\right)^{1/4} \qquad \text{is greater than 1.5.}$$

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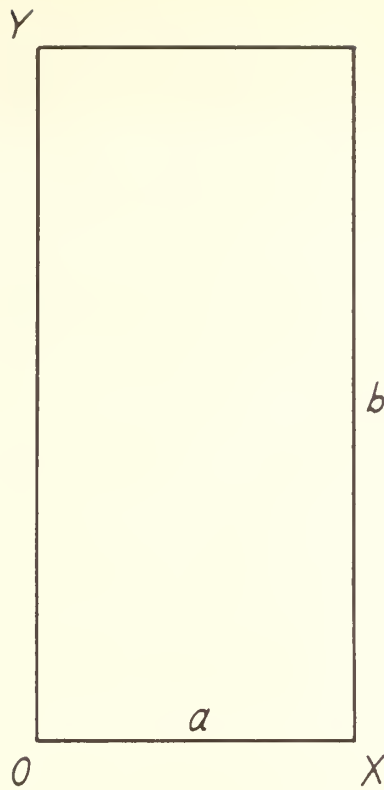
FIG. 1
SIDE AND AXES DESIGNATIONS
FOR A RECTANGULAR PLATE
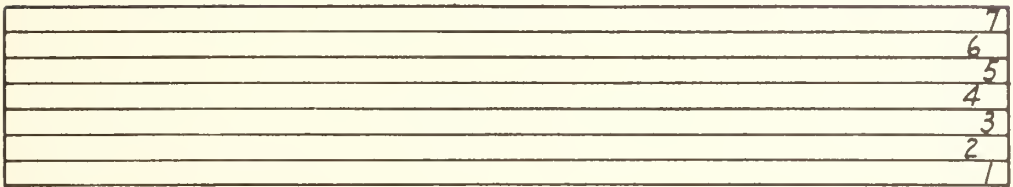
FIG. 2
SECTION OF A PLYWOOD STRIP
SHOWING NUMBERING OF PLIES
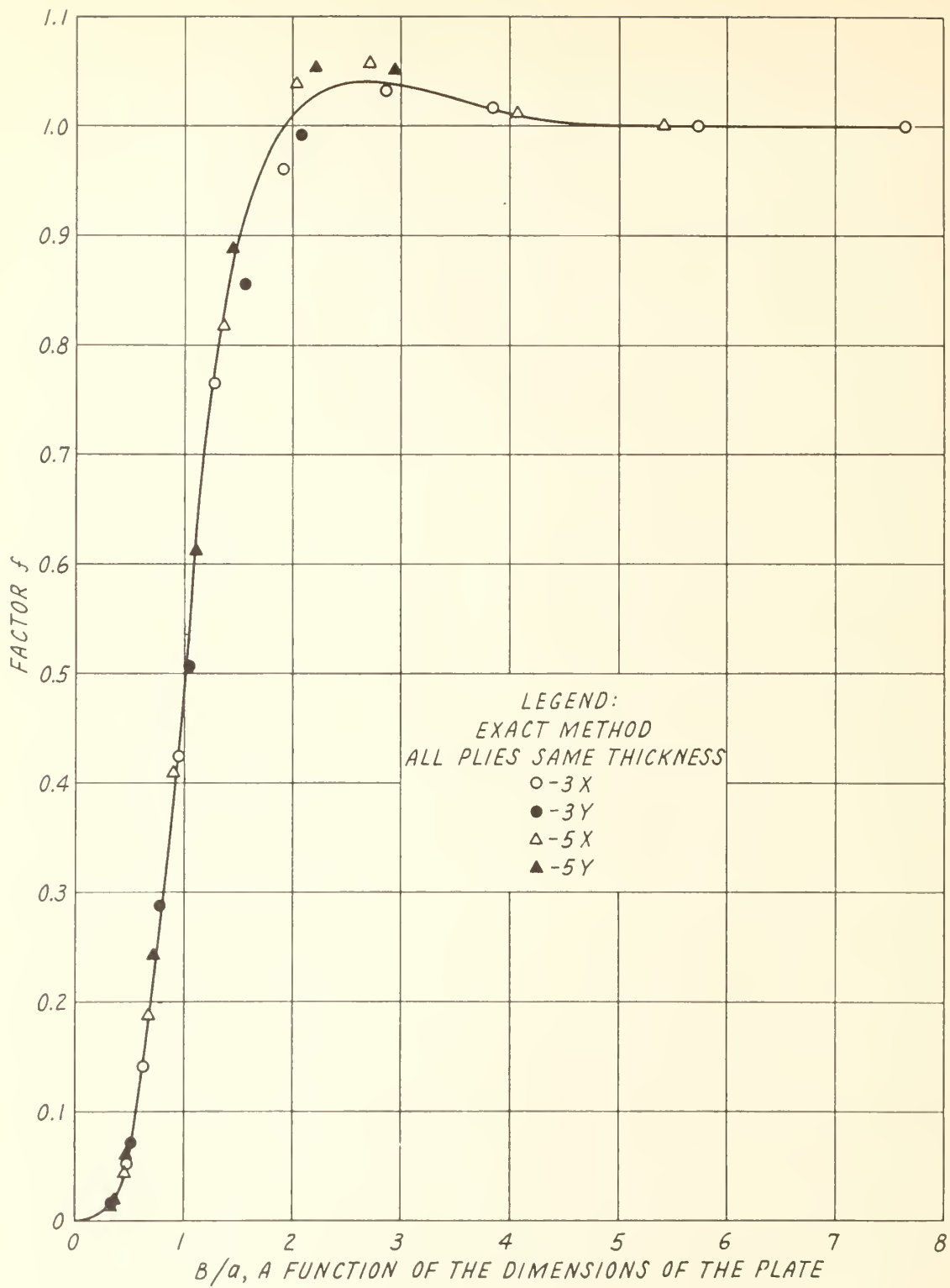
FIG. 3
FACTOR f [FORMULA (2)] AS A FUNCTION OF B/a, WHERE $B = b\left(E_1/E_2\right)^{\frac{1}{4}}$.
UNIFORM LOAD. EDGES SIMPLY SUPPORTED.

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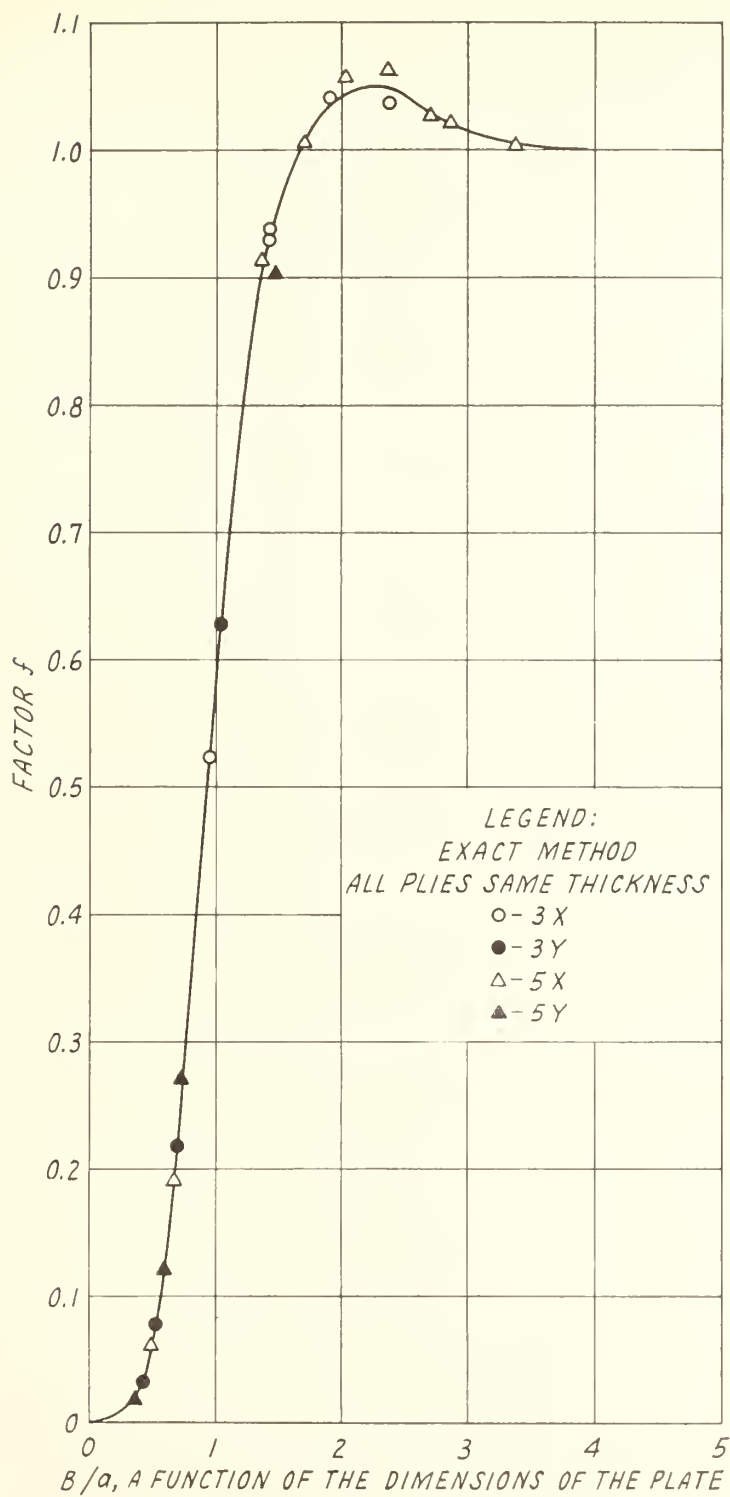
FIG. 4
FACTOR f [FORMULA (5)] AS A FUNCTION OF B/a,
WHERE $B = b\left(E_1/E_2\right)^{\frac{1}{4}}$. UNIFORM LOAD. EDGES CLAMPED.

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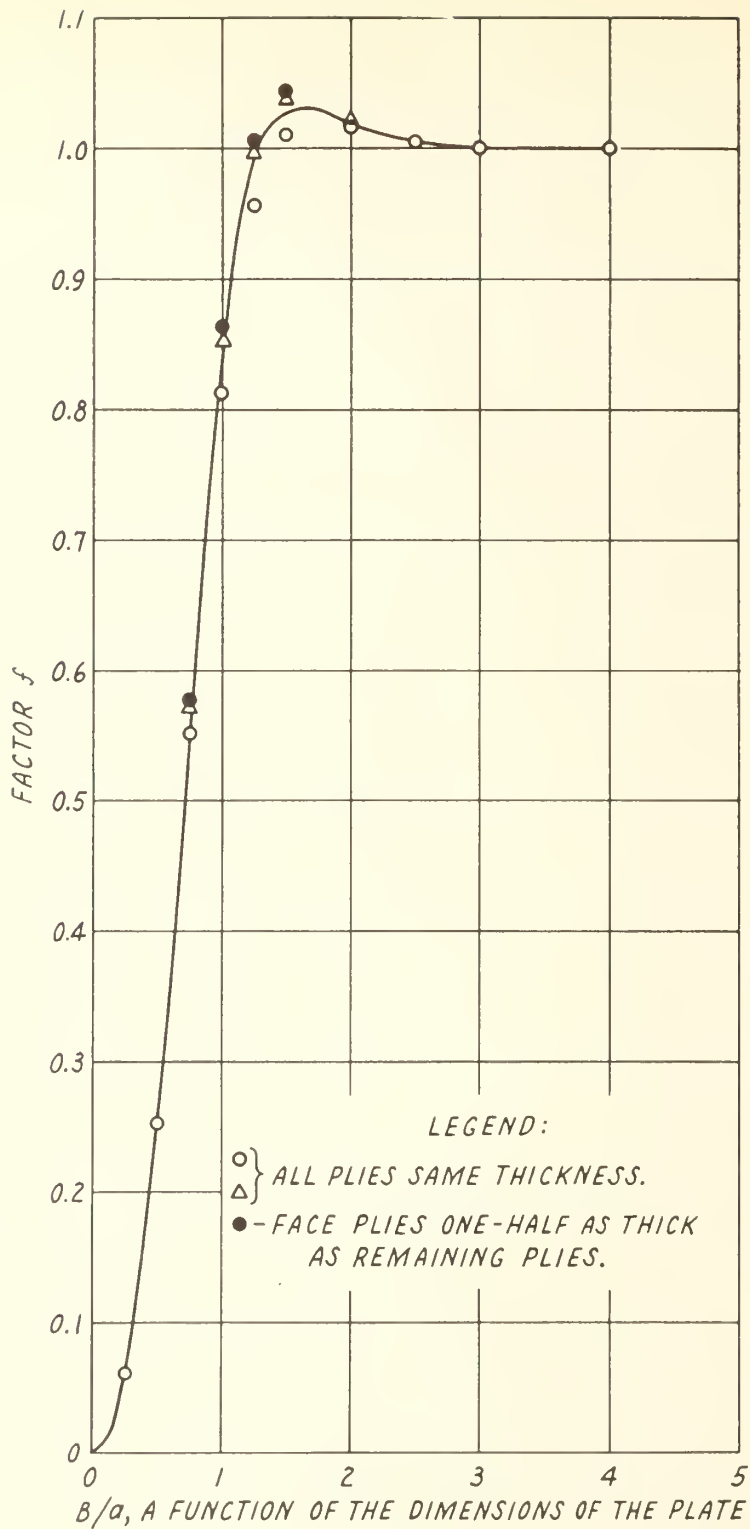
FIG. 5
FACTOR f [FORMULA (8)] AS A FUNCTION OF B/a, WHERE
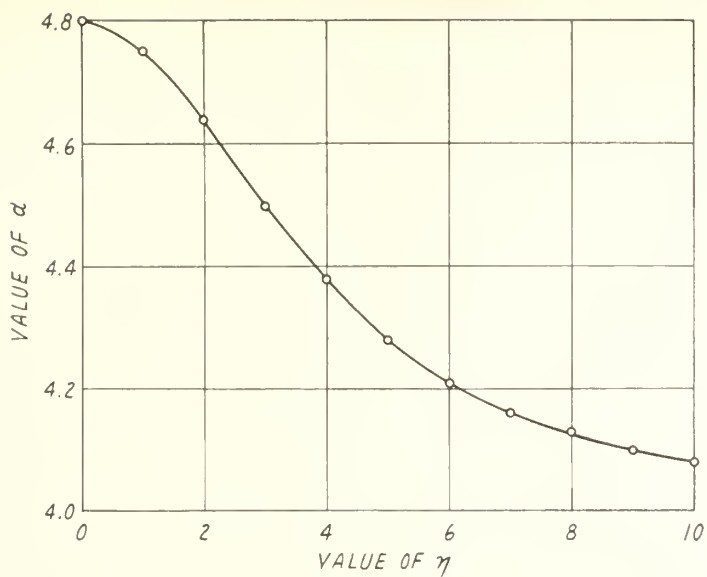$B = b(E_1/E_2)^{\frac{1}{4}}$. CONCENTRATED LOAD. EDGES SIMPLY SUPPORTED.

FIG. 6
THE COEFFICIENT α IN THE FORMULA (14)
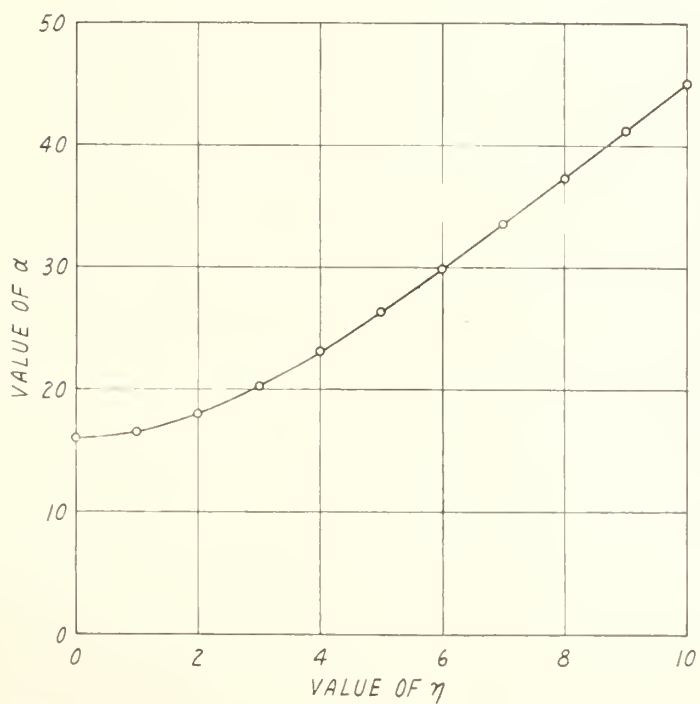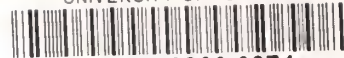AS A FUNCTION OF η

FIG. 7
THE COEFFICIENT α IN THE FORMULA (20)
AS A FUNCTION OF η

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